-: HAMILTON'S EQUATIONS: -

The Hamiltonian is a function of generalised Coordinates q_K, generalised momentum P_K and time t, i.e

$$H = H(q_1, q_2 \cdots q_K, \cdots q_n, P_1, P_2, \cdots P_K \cdots P_n, t) \quad \text{(1)}$$

The differentiation dH as,

$$dH = \sum_K \frac{\partial H}{\partial q_K} dq_K + \sum_K \frac{\partial H}{\partial P_K} dP_K + \frac{\partial H}{\partial t} dt \quad \text{(2)}$$

But, $H = \sum_K P_K \dot{q}_K - L$

Hence, $dH = \sum_K \dot{q}_K dP_K + \sum_K P_K d\dot{q}_K - dL \quad \text{(3)}$

also, $L = L(q_1, q_2, \cdots q_K, \cdots q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_K, \cdots \dot{q}_n, t)$

Therefore, $dL = \sum_K \frac{\partial L}{\partial q_K} dq_K + \sum_K \frac{\partial L}{\partial \dot{q}_K} d\dot{q}_K + \frac{\partial L}{\partial t} dt$

But, $\dot{P}_K = \frac{\partial L}{\partial q_K}$ and $P_K = \frac{\partial L}{\partial \dot{q}_K}$

therefore, $dL = \sum_K \dot{P}_K dq_K + \sum_K P_K d\dot{q}_K + \frac{\partial L}{\partial t} dt \quad \text{(4)}$

Substituting for dL from (4) in eqn (3), we get

$$dH = \sum_K \dot{q}_K dP_K - \sum_K \dot{P}_K dq_K - \frac{\partial L}{\partial t} dt \quad \text{(5)}$$

Comparing the Coefficients of 'dP_K', 'dq_K' and 'dt' in eqn (1) and (4), we obtain,

$$\dot{q}_K = \frac{\partial H}{\partial p_K} \quad \text{———} \quad (6)$$

$$-\dot{p}_K = \frac{\partial H}{\partial q_K} \quad \text{———} \quad (7)$$

$$\text{and } -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{———} \quad (8)$$

eqn. (6) & (7) are known as Hamilton's equations or Hamilton's Canonical equations of motion. This procedure of the motion of a system by these equations is called Hamiltonian dynamics.

For, $K = 1, 2, \ldots n$ in all these are $2n$ first order differential equations which are much easier to solve in comparison to the n second order differential equations in Lagrangian dynamics.

from eqn (7), If any coordinate q_K is cyclic, i.e not contained in H, then

$$\frac{\partial H}{\partial q_K} = 0, \text{ or, } \dot{p}_K = 0 \text{ or, } p_K = \text{Constant in time.}$$

from eqn (1), $$\frac{dH}{dt} = \sum \frac{\partial H}{\partial q_K}\dot{q}_K + \sum_K \frac{\partial H}{\partial p_K}\dot{p}_K + \frac{\partial H}{\partial t} \quad \text{—} \quad (9)$$

Substituting for $\dot{q}_K$ and $\dot{p}_K$ from eqn (6) & (9), we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \text{—} \quad (10)$$

If the Langrangian L and hence H does not depend on time t explicitly, then $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = 0$ and

$$\frac{dH}{dt} = 0 \quad \text{or,} \quad \boxed{H = \text{Constant}}$$

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